ELSEVIER
Journal of Pure and Applied Algebra 152 (2000) 93–107
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Cohen–Macaulayness and negativity of A-invariants in Rees algebras associated to $\mathfrak{m}$-primary ideals of minimal multiplicity

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Received 21 September 1998; received in revised form 13 May 1999

Abstract

Let I be an $\mathfrak{m}$-primary ideal in a Cohen–Macaulay local ring $(A,\mathfrak{m})$ of $d=\dim A \geq 1$. The ideal I is said to have minimal multiplicity if $\mu_A(I)=\mathrm{e}_I(A)+d-\ell_A(A/I)$. There are given criteria for the Cohen–Macaulayness and Gorensteinness in Rees algebras $\mathrm{R}(I)$ and graded rings $\mathrm{G}(I)$ associated to $\mathfrak{m}$-primary ideals I of minimal multiplicity. The Cohen–Macaulayness in $\mathrm{R}(I)$ is explored in connection with that of $\operatorname{Proj}\mathrm{R}(I)$ and the negativity of invariants $\mathrm{a}_i(\mathrm{R}(I))$. A counterexample to a conjecture of Korb and Nakamura will be given.

MSC: Primary 13A30; secondary 13H10

1. Introduction

In this paper, I survey the theory developed in [6] for the Cohen–Macaulayness in Rees algebras and graded rings associated to a certain class of ideals in Cohen–Macaulay local rings. Researches in [6] originated from the analysis of the Buchsbaumness in Rees algebras and graded rings associated to the class of ideals. But in this paper let me focus my attention mainly on the Cohen–Macaulay property.

[1] The author is supported by the Grant-in-Aid for Scientific Researches (C) (1997–1998, No. 09640071).
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PII: S0022-4049(99)00137-1

Let A be a Cohen–Macaulay local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A \geq 1$. Assume that the field $A/\mathfrak{m}$ is infinite. Let t be an indeterminate over A. For an ideal $I(\neq A)$ in A we define

$$\mathrm{R}(I) = A[It] \subseteq A[t],$$

$$\mathrm{R}'(I) = A[It, t^{-1}] \subseteq A[t, t^{-1}],$$

$$\mathrm{G}(I) = \mathrm{R}'(I)/t^{-1}\mathrm{R}'(I)$$

which we call the Rees algebra, the extended Rees algebra, and the associated graded ring of I respectively. As is well-known, the canonical morphism $\operatorname{Proj} \mathrm{R}(I) \to \operatorname{Spec} A$, that is the blowing-up of A with the center I plays a very important role in the analysis of singularities. In this paper we shall also explore the Cohen–Macaulayness of $\operatorname{Proj} \mathrm{R}(I)$, but our interest lies mostly in the analysis of the structure of algebras $\mathrm{R}(I)$.

Let I be an $\mathfrak{m}$-primary ideal in A and let Q be a minimal reduction of I. Hence Q is generated by d elements and $Q \subseteq I$ with $I^{n+1} = QI^n$ [15]. Let $\mathrm{e}_I(A)$ denote the multiplicity of A with respect to I. Then we have the inequality

$$\mu_A(I) \leq \mathrm{e}_I(A) + d - \ell_A(A/I)$$

(here for a given A-module $E, \mu_A(E)$ and $\ell_A(E)$ denote the number of elements in a minimal system of generators for E and the length of E, respectively), in which the equality is attained if and only if $\mathfrak{m}I = \mathfrak{m}Q$, or equivalently $\mathfrak{m}I \subseteq Q$ [2]. We say that the ideal I has minimal multiplicity if the equality $\mu_A(I) = \mathrm{e}_I(A) + d - \ell_A(A/I)$ holds. Therefore a Cohen–Macaulay local ring A possesses maximal embedding dimension in the sense of Sally [16] if and only if the maximal ideal $\mathfrak{m}$ of A has minimal multiplicity.

In this paper firstly we shall explore the Cohen–Macaulay and Gorenstein properties of graded algebras $\mathrm{R}(I)$ and $\mathrm{G}(I)$ associated to $\mathfrak{m}$-primary ideals I of minimal multiplicity, which we will perform in Section 2. Here let us summarize the results on the Cohen–Macaulayness in $\mathrm{R}(I)$ into the following.

Theorem 2.7. *Suppose $d = \dim A \geq 2$ and let I be an $\mathfrak{m}$-primary ideal in A possessing minimal multiplicity. Let Q be a minimal reduction of I. Then the following four conditions are equivalent.*
(1) $\mathrm{R}(I)$ *is a Cohen–Macaulay ring.*
(2) $\mathrm{G}(I)$ *is a Cohen–Macaulay ring.*
(3) *The fiber cone* $\mathrm{S}(I) = A/\mathfrak{m} \otimes_A \mathrm{R}(I)$ *is a Cohen–Macaulay ring possessing maximal embedding dimension.*
(4) $I^2 = QI$.
When this is the case, for all integers $n \geq 0$ we have the equalities

$$\mu_A(I^n) = \binom{d+n-1}{d-1} + m\binom{d+n-2}{d-1},$$

$$\ell_A(A/I^{n+1}) = \ell\binom{d+n}{d} + m\binom{d+n-1}{d},$$

where $\ell = \ell_A(A/I)$ and $m = \ell_A(I/Q) = \mu_A(I) - d$.

In Section 3, we will give some basic examples of $\mathfrak{m}$-primary ideals of minimal multiplicity. As suggested by Korb and Nakamura in [14], at least in the case where $\dim A$ is small, the negativity of invariants $\mathrm{a}_i(\mathrm{R}(I))$'s of $\mathrm{R}(I)$ has some influence on the Cohen–Macaulayness in $\mathrm{R}(I)$. Secondly, in Section 4, we shall explore this phenomenon in our context. So, we now briefly recall the definition of a-invariant.

For a moment let $R = \bigoplus_{n\in\mathbb{Z}} R_n$ be a Noetherian graded ring and assume that R contains a unique graded maximal ideal, say M. We denote by $\mathrm{H}^i_M(*)$ $(i\in\mathbb{Z})$ the ith local cohomology functor of R relative to M. For a given graded R-module E and $n\in\mathbb{Z}$, let $[\mathrm{H}^i_M(E)]_n$ denote the homogeneous component of the graded R-module $\mathrm{H}^i_M(E)$ with degree n. Then if $R_n=(0)$ and $n<0$ and E a finitely generated graded R-module, we have $[\mathrm{H}^i_M(E)]_n=(0)$ for all $n\gg 0$ and $i\in\mathbb{Z}$; so, we define

$$\mathrm{a}_i(E)=\sup\{n\in\mathbb{Z}\mid [\mathrm{H}^i_M(E)]_n\neq(0)\}$$

and call it the ith a-invariant of E. When $\dim_R E=s$, we denote $\mathrm{a}_s(E)$ simply by $\mathrm{a}(E)$ and call it the a-invariant of E (cf. [9, (3.1.4)]).

With the notation introduced above, among others we will prove in Section 4 the following, in which $\mathrm{r}(A):=\ell_A(\mathrm{Ext}^d_A(A/\mathfrak{m},A))$ denotes the Cohen–Macaulay type of A.

Corollary 4.8. *Let I be an $\mathfrak{m}$-primary ideal in A possessing minimal multiplicity. Suppose that $\dim A=3$ and $\mathrm{r}(A)\le 3$. Then $\mathrm{R}(I)$ is Cohen–Macaulay ring if and only if $\mathrm{Proj}\,\mathrm{R}(I)$ is a Cohen–Macaulay scheme and $\mathrm{a}_i(\mathrm{R}(I))<0$ for all $i\in\mathbb{Z}$.*

In Section 5, we explore an Example (Example 5.1), which shows the hypothesis in Corollary 4.8 that $\mathrm{r}(A)\le 3$ is not superfluous. It provides the main conjecture of Korb and Nakamura [14] with a counterexample, where they asked whether the ring $\mathrm{R}(I)$ is Cohen–Macaulay once $\mathrm{Proj}\,\mathrm{R}(I)$ is a Cohen–Macaulay scheme and $\mathrm{a}_i(\mathrm{R}(I))<0$ for all $i\in\mathbb{Z}$. The example is a by-product of the research in [6].

In what follows, let A denote a Cohen–Macaulay local ring with the maximal ideal $\mathfrak{m}$ and $d=\dim A\ge 1$. We assume that the field $A/\mathfrak{m}$ is infinite. Let $\mathrm{H}^i_{\mathfrak{m}}(*)$ $(i\in\mathbb{Z})$ stand for the ith local cohomology functor of A with respect to $\mathfrak{m}$. Unless otherwise specified, for a given Noetherian graded ring $R=\bigoplus_{n\in\mathbb{Z}}R_n$ with a unique graded maximal ideal M and a finitely generated graded R-module E, we shall simply denote $\dim_{R_M}E_M$ and $\mathrm{depth}_{R_M}E_M$ by $\dim_R E$ and $\mathrm{depth}_R E$, respectively.

2. Cohen–Macaulayness and Gorensteinness in R(I)

Let I be an $\mathfrak{m}$-primary ideal in A and let $Q=(a_1,a_2,\ldots,a_d)$ be a minimal reduction of I. Let $\mathscr{R}=\mathrm{R}(I)$, $\mathscr{R}'=\mathrm{R}'(I)$, $\mathscr{G}=\mathrm{G}(I)$, and $\mathscr{S}=A/\mathfrak{m}\otimes_{A/\mathfrak{m}}\mathscr{R}$. Hence $\dim\mathscr{R}=\dim\mathscr{R}'=d+1$ and $\dim\mathscr{G}=\dim\mathscr{S}=d$. Let M denote the unique graded maximal ideal in $\mathscr{R}$ and let $f_i=a_it$ for $1\le i\le d$. The purpose of this section is to explore the Cohen–Macaulayness and Gorensteinness in $\mathscr{R}$ and $\mathscr{G}$.

We begin with the following. This is known by Chuai [2] but let me give a brief proof for completeness.

Lemma 2.1 (Chuai [2]). *The following assertions hold true:*
(1) $\mu_A(I) \leq \mathrm{e}_I(A) + d - \ell_A(A/I)$.
(2) $\mu_A(I) = \mathrm{e}_I(A) + d - \ell_A(A/I)$ *if and only if* $\mathfrak{m}I = \mathfrak{m}Q$.

Proof. Recall that $\mathrm{e}_I(A) = \mathrm{e}_Q(A) = \ell_A(A/Q)$ and that $\mathfrak{m}I \cap Q = \mathfrak{m}Q$ [15]. Let $E = I/Q$. Then by the standard exact sequence $0 \to Q/\mathfrak{m}Q \to I/\mathfrak{m}I \to E/\mathfrak{m}E \to 0$ we have $\mu_A(I) = d + \mu_A(E)$. Hence $\mu_A(I) \leq \mathrm{e}_I(A) + d - \ell_A(A/I)$, because $\mu_A(E) \leq \ell_A(E)$ and $\ell_A(E) = \ell_A(A/Q) - \ell_A(A/I) = \mathrm{e}_I(A) - \ell_A(A/I)$. The equality $\mu_A(I) = \mathrm{e}_I(A) + d - \ell_A(A/I)$ holds if and only if $\mu_A(E) = \ell_A(E)$, that is $\mathfrak{m}I \subseteq Q$, or equivalently $\mathfrak{m}I = \mathfrak{m}Q$. □

We now assume that our ideal I has minimal multiplicity. Then $\mathfrak{m}I = \mathfrak{m}Q$ by Lemma 2.1 and so the equality $\mathfrak{m}I^n = \mathfrak{m}Q^n$ holds true for all $n \in \mathbb{Z}$, whence $\mathfrak{m}\mathscr{R} = \mathfrak{m}\mathrm{R}(Q)$. Let $\mathscr{C} = \mathscr{R}/\mathrm{R}(Q)$ (hence $\mathscr{C}_n = (0)$ if $n \leq 0$). Then as $\mathfrak{m}\mathscr{C} = (0)$, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{R}(Q) & \longrightarrow & \mathscr{R} & \longrightarrow & \mathscr{C} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathscr{P} & \longrightarrow & \mathscr{S} & \longrightarrow & \mathscr{C} & \longrightarrow & 0 \end{array} \tag{2.1}$$

with exact rows, in which $\mathscr{P} = A/\mathfrak{m} \otimes_A \mathrm{R}(Q)$ and the vertical maps are canonical epimorphisms. Recall that $\mathscr{P}$ is a polynomial ring in d variables over the field $A/\mathfrak{m}$ and that $\mathscr{S}$ is a module-finite extension of $\mathscr{P}$. Then by a theorem of Hochster [10] the bottom row in (2.1) is split, and so we have

$$\mathscr{S} \cong \mathscr{P} \oplus \mathscr{C} \tag{2.2}$$

as graded $\mathscr{P}$-modules. Let N be the unique graded maximal ideal in $\mathrm{R}(Q)$. Then as $\mathscr{P}$ is a polynomial ring, by (2.2) we get $\mathrm{H}^i_M(\mathscr{S}) \cong \mathrm{H}^i_N(\mathscr{C})$ for $i \leq d-1$ and $\mathrm{H}^d_M(\mathscr{S}) \cong \mathrm{H}^d_N(\mathscr{P}) \oplus \mathrm{H}^d_N(\mathscr{C})$. On the other hand, since $\mathrm{R}(Q)$ is a Cohen–Macaulay ring of $\dim \mathrm{R}(Q) = d+1$, from the top row in (2.1) the isomorphisms $\mathrm{H}^i_M(\mathscr{R}) \cong \mathrm{H}^i_N(\mathscr{C})$ for $i \leq d-1$ and the exact sequence $0 \to \mathrm{H}^d_M(\mathscr{R}) \to \mathrm{H}^d_N(\mathscr{C}) \to \mathrm{H}^{d+1}_N(\mathrm{R}(Q))$ follow. Because $\mathrm{a}(\mathrm{R}(Q)) = -1$ and $\mathrm{a}(\mathscr{P}) = -d$ (cf. [7, Part II, (3.3); 9, (3.1.6)]), summarizing these observations, we have the following.

Proposition 2.2. (1) $[\mathrm{H}^i_M(\mathscr{R})]_n \cong [\mathrm{H}^i_N(\mathscr{C})]_n \cong [\mathrm{H}^i_M(\mathscr{S})]_n$ *for all* $i \leq d-1$ *and* $n \in \mathbb{Z}$.
(2) $\mathrm{a}_i(\mathscr{R}) = \mathrm{a}_i(\mathscr{C}) = \mathrm{a}_i(\mathscr{S})$ *if* $i \leq d-1$.
(3) $\mathrm{a}(\mathscr{S}) = \max\{\mathrm{a}_d(\mathscr{C}), -d\}$.
(4) $\mathrm{a}_d(\mathscr{R}) \leq \mathrm{a}_d(\mathscr{C}) \leq \max\{\mathrm{a}_d(\mathscr{R}), -1\}$.

Lemma 2.3. (1) *Let* $I \neq Q$. *Then* $\dim_{\mathscr{P}} \mathscr{C} = d$ *and* $\min\{\operatorname{depth}_{\mathscr{P}} \mathscr{C}, \operatorname{depth} \mathscr{S}\} \geq 1$.
(2) $\mathscr{G}$ *is a Cohen–Macaulay ring if and only if* $I^2 = QI$. *When this is the case,* $\mathrm{a}(\mathscr{G}) = 1 - d$ *if* $I \neq Q$ *and* $\mathrm{a}(\mathscr{G}) = -d$ *if* $I = Q$.
(3) $\operatorname{depth} \mathscr{R} = \operatorname{depth} \mathscr{G} + 1$ *if* $d \geq 2$.

Proof. (1) We have $\mathscr{C} \neq (0)$ since $I \neq Q$. As $\mathrm{H}^0_M(\mathscr{R}) = (0)$, by Proposition 2.2(1) we get $\mathrm{H}^0_N(\mathscr{C}) = \mathrm{H}^0_M(\mathscr{S}) = (0)$, whence $\min\{\operatorname{depth}_{\mathscr{P}} \mathscr{C}, \operatorname{depth} \mathscr{S}\} \geq 1$. The element f_d is actually $\mathscr{C}$-regular. In fact, let $x \in I^n$ with $n \geq 1$ and assume $f_d \cdot xt^n \in \mathrm{R}(Q)$. Then $a_d x \in a_d A \cap Q^{n+1} = a_d Q^n$ whence $x \in Q^n$. To see that $\dim_{\mathscr{P}} \mathscr{C} = d$, it suffices to check $\dim_{\mathscr{P}} \mathscr{C} \geq d$. This is clear for $d = 1$. Let $d \geq 2$ and assume that our assertion is true for $d-1$. Let $\overline{A} = A/a_d A$, $\overline{\mathfrak{m}} = \mathfrak{m}/a_d$, $\overline{I} = I/a_d A$, and $\overline{Q} = Q/a_d A$. Then $\overline{Q}$ is a reduction of $\overline{I}$ with $\overline{\mathfrak{m}}\overline{I} = \overline{\mathfrak{m}}\overline{Q}$. Hence the ideal $\overline{I}$ has minimal multiplicity, and so, from the hypothesis on d we see $\dim_{\overline{P}} \mathrm{C}(\overline{I}) \geq d-1$, where $\mathrm{C}(\overline{I}) = \mathrm{R}(\overline{I})/\mathrm{R}(\overline{Q})$ and $\overline{\mathscr{P}} = \overline{A}/\overline{\mathfrak{m}} \otimes_{\overline{A}} \mathrm{R}(\overline{Q})$. As $\mathrm{C}(\overline{I})$ is naturally a homomorphic image of $\mathscr{C}/f_d\mathscr{C}$, we get $\dim_{\mathscr{P}} \mathscr{C}/f_d\mathscr{C} \geq d-1$. Thus $\dim_{\mathscr{P}} \mathscr{C} \geq d$ since f_d is $\mathscr{C}$-regular.

(2) The *if part* is due to [19, (3.1)]. Let $\mathscr{G}$ be a Cohen–Macaulay ring. Then $Q \cap I^n = QI^{n-1}$ for all $n \in \mathbb{Z}$ by [19, (2.7)], while $I^2 \subseteq Q$ as $I^2 \subseteq \mathfrak{m}I = \mathfrak{m}Q$. Hence $I^2 = QI$. The last assertion now follows from the equality $\mathrm{a}(\mathscr{G}) = \mathrm{a}(\mathscr{G}/(f_1, f_2, \ldots, f_d)\mathscr{G}) - d$ (cf. [9, (3.1.6)]), because $\mathrm{a}(\mathscr{G}/(f_1, f_2, \ldots, f_d)\mathscr{G}) = 1$ (resp. $\mathrm{a}(\mathscr{G}/(f_1, f_2, \ldots, f_d)\mathscr{G}) = 0$) if $I \neq Q$ (resp. $I = Q$).

(3) If $\mathscr{G}$ is Cohen–Macaulay, then $I^2 = QI$ by (2) and so the ring $\mathscr{R}$ is Cohen–Macaulay (cf. [8, (3.10)]). The equality $\operatorname{depth} \mathscr{R} = \operatorname{depth} \mathscr{G} + 1$ is due to [12] in the case where $\mathscr{G}$ is not Cohen–Macaulay. □

If the ring $\mathscr{S}$ is not Cohen–Macaulay, then we have $I \neq Q$ and $\operatorname{depth} \mathscr{R} = \operatorname{depth} \mathscr{S} = \operatorname{depth}_{\mathscr{P}} \mathscr{C}$ as well (cf. Proposition 2.2(1)). As $d \geq 2$ by parts (1) and (3) of Lemma 2.3 we see $\operatorname{depth} \mathscr{R} = \operatorname{depth} \mathscr{G} + 1$. Hence

Corollary 2.4. *Suppose that $\mathscr{S}$ is not a Cohen–Macaulay ring. Then* $\operatorname{depth} \mathscr{R} = \operatorname{depth} \mathscr{S} = \operatorname{depth}_{\mathscr{P}} \mathscr{C} = \operatorname{depth} \mathscr{G} + 1 < d$.

Let $\mathrm{r}_Q(I) = \min\{n \geq 0 \mid I^{n+1} = QI^n\}$ and call it the reduction number of I with respect to Q.

Proposition 2.5. *The following four conditions are equivalent*:
(1) *$\mathscr{S}$ is a Cohen–Macaulay ring.*
(2) $\operatorname{depth} \mathscr{R} \geq d$.
(3) $\operatorname{depth} \mathscr{G} \geq d-1$.
(4) *$\mathscr{C}$ is $\mathscr{P}$-free.*
When this is the case, $\mathrm{a}(\mathscr{S}) = \mathrm{r}_Q(I) - d$.

Proof. (1) ⇔ (2) ⇔ (4): See Proposition 2.2(1).

(2) ⇔ (3): This follows from Lemma 2.3(3).

To check the last equality, note $\mathrm{a}(\mathscr{S}) = \mathrm{a}(\mathscr{S}/(f_1, f_2, \ldots, f_d)\mathscr{S}) - d$ (cf. [9, (3.1.6)]). Then as $\mathrm{a}(\mathscr{S}/(f_1, f_2, \ldots, f_d)\mathscr{S}) = \max\{n \geq 0 \mid I^n \nsubseteq QI^{n-1} + \mathfrak{m}I^n\}$, via Nakayama's lemma we get $\mathrm{a}(\mathscr{S}/(f_1, f_2, \ldots, f_d)\mathscr{S}) = \mathrm{r}_Q(I)$. Thus $\mathrm{a}(\mathscr{S}) = \mathrm{r}_Q(I) - d$. □

Corollary 2.6. *$\mathscr{G}$ is a Cohen–Macaulay ring if and only if $\mathscr{S}$ is a Cohen–Macaulay ring and* $\mathrm{a}(\mathscr{S}) \leq 1-d$.

Proof. See Lemma 2.3(2) and Proposition 2.5. □

When $d=1$, $\mathscr{R}$ is a Cohen–Macaulay ring if and only if $I=Q$ (cf. [8, (3.10)]. As for the case where $d \geq 2$, we note the following.

Theorem 2.7. *Suppose $d \geq 2$. Then the following four conditions are equivalent*:
(1) *$\mathscr{R}$ is a Cohen–Macaulay ring.*
(2) *$\mathscr{G}$ is a Cohen–Macaulay ring.*
(3) *$\mathscr{S}$ is a Cohen–Macaulay ring possessing maximal embedding dimension.*
(4) *$I^2 = QI$.*
When this is the case, for all $n \geq 0$ we have the equalities

$$\mu_A(I^n) = \binom{d+n-1}{d-1} + m\binom{d+n-2}{d-1},$$

$$\ell_A(A/I^{n+1}) = \ell\binom{d+n}{d} + m\binom{d+n-1}{d},$$

where $\ell = \ell_A(A/I)$ and $m = \ell_A(I/Q) = \mu_A(I) - d$.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4): This follows from parts (2) and (3) of Lemma 2.3.

(2) $\Leftrightarrow$ (3): By Proposition 2.5 we may assume $\mathscr{S}$ is Cohen–Macaulay. Let $\mathfrak{M} = \mathscr{S}_+$. Then $\mathscr{S}$ has maximal embedding dimension if and only if $\mathfrak{M}^2 = (f_1, f_2, \ldots, f_d)\mathfrak{M}$, and the latter condition is equivalent to saying that $\mathrm{a}(\mathscr{S}/(f_1, f_2, \ldots, f_d)\mathscr{S}) \leq 1$. So the equivalence (2) $\Leftrightarrow$ (3) follows from Corollary 2.6, since $\mathrm{a}(\mathscr{S}/(f_1, f_2, \ldots, f_d)\mathscr{S}) = a(\mathscr{S}) + d$.

See [17, Theorem 6] for the last two equalities. $\square$

The next result directly follows from [4, (2.1), (3.1) and (3.2)]. Let me give a brief proof for completeness.

Corollary 2.8. *Suppose that A is a Gorenstein ring. Then $\mathscr{G}$ is a Cohen–Macaulay ring. If $d \geq 2$, $\mathscr{R}$ is a Cohen–Macaulay ring too.*

Proof. We may assume $I \neq Q$. Then we have $I = Q : \mathfrak{m}$, because $Q \subseteq I \subseteq Q : \mathfrak{m}$ and $\ell_A((Q : \mathfrak{m})/Q) = 1$. We will show $I^2 = QI$. Let $a, b \in I$ and write $ab = \sum_{i=1}^d a_i c_i$ with $c_i \in A$. Then for each $x \in \mathfrak{m}$ we get $xab \in Q^2$ (since $\mathfrak{m}I^2 = \mathfrak{m}Q^2 \subseteq Q^2$). Therefore $\sum_{i=1}^d a_i \cdot xc_i \in Q^2$ and so $xc_i \in Q$ for all $1 \leq i \leq d$. Hence $c_i \in Q : \mathfrak{m} = I$ so that $I^2 = QI$. The last assertion follows from Theorem 2.7. $\square$

Let us add a few remarks on the Gorenstein property of $\mathscr{R}$ and $\mathscr{G}$.

Proposition 2.9. *Suppose $I^2 = QI$. Then the Cohen–Macaulay type $\mathrm{r}(\mathscr{G})$ of $\mathscr{G}$ is given by the following formula*:

$$\mathrm{r}(\mathscr{G}) = \begin{cases} \mathrm{r}(A/I) + \mu_A(I) - d & \text{if } I \neq \mathfrak{m}, \\ \mu_A(I) - d & \text{if } I = \mathfrak{m} \neq Q, \\ 1 & \text{if } I = \mathfrak{m} = Q. \end{cases}$$

Proof. Recall $\mathrm{r}(\mathscr{G}) = \mathrm{r}(\mathscr{G}_{\mathfrak{M}})$ (cf. [1]) where $\mathfrak{M}$ is the graded maximal ideal in $\mathscr{G}$. On the other hand, since the sequence $f_1, f_2, \dots, f_d$ is $\mathscr{G}$-regular (cf. [19, (2.1)]), we have $\mathrm{r}(\mathscr{G}_{\mathfrak{M}}) = \mathrm{r}(\mathscr{G}/(f_1, f_2, \dots, f_d)\mathscr{G})$ and the isomorphism $\mathscr{G}/(f_1, f_2, \dots, f_d)\mathscr{G} = \mathrm{G}(I/Q)$ as well. Thus $\mathrm{r}(\mathscr{G}) = \mathrm{r}(\mathrm{G}(I/Q))$. Let V denote the socle of $\mathrm{G}(I/Q) = A/I \oplus I/Q$. Then $V = (I : \mathfrak{m})/I \oplus I/Q$ if $I \neq \mathfrak{m}$, $V = I/Q$ if $I = \mathfrak{m} \neq Q$, and $V = A/\mathfrak{m}$ if $I = \mathfrak{m} = Q$, from which the formula follows because $\ell_A(I/Q) = \mu_A(I) - d$. $\square$

The following two corollaries are particular cases of [3, Corollaries 3.2 and 3.3].

Corollary 2.10. *$\mathscr{G}$ is a Gorenstein ring if and only if either* (1) *$I = Q$ and A is a Gorenstein ring or* (2) *$I = \mathfrak{m}$ and $\mu_A(\mathfrak{m}) = d + 1$.*

Corollary 2.11. *$\mathscr{R}$ is a Gorenstein ring if and only if either* (1) *$d \leq 2$, $I = Q$, and A is a Gorenstein ring or* (2) *$d = 3$, $I = \mathfrak{m}$, and $\mu_A(\mathfrak{m}) = 4$.*

3. Basic examples

In this section we note some basic examples of $\mathfrak{m}$-primary ideals of minimal multiplicity. Throughout let k denote an infinite field.

A typical example satisfying condition (2) of Corollary 2.11 is as follows.

Example 3.1. Let $R = k[[X, Y, Z, W]]$ be the formal power series ring and $A = R/(XY - ZW)$. Let x, y, z, and w denote, respectively, the reductions of X, Y, Z, and W mod$(XY - ZW)$. Then $\dim A = 3$ and $\mathfrak{m}^2 = (x, y, z - w)\mathfrak{m}$. The maximal ideal $\mathfrak{m}$ in A has minimal multiplicity with $\mu_A(\mathfrak{m}) = 4$ and $\mathrm{R}(\mathfrak{m})$ is a Gorenstein ring.

If every $\mathfrak{m}$-primary ideal in A has minimal multiplicity, $\dim A = 1$ and A is reduced. In fact, assume $d \geq 2$ and let $a_1, a_2, \dots, a_d$ be a system of parameters for A. Let $Q = (a_1^4, a_2^4, a_3, \dots, a_d)$ and let $\overline{Q}$ denote the integral closure of Q. Then $a_1^2 a_2^2 \in \overline{Q}$ but $a_1^3 a_2^3 \notin Q$, so that $\overline{Q}$ does not have minimal multiplicity. Hence $d = 1$. Let $N = \sqrt{(0)}$ and take a nonzerodivisor $a \in \mathfrak{m}$. Then $\mathfrak{m}N \subseteq a^n A$ for all $n \geq 1$ as $N \subseteq \overline{a^n A}$. Therefore $\mathfrak{m}N = (0)$ and A is a reduced ring.

We have the following characterization (Proposition 3.2) of one-dimensional Cohen–Macaulay local rings in which every $\mathfrak{m}$-primary ideal has minimal multiplicity. To state the result let $\overline{A}$ denote the integral closure of A in its total quotient ring. Let $\mathfrak{c} = A : \overline{A}$ be the conductor. We put $\mathrm{v}(A) = \mu_A(\mathfrak{m})$ and $\mathrm{e}(A) = \mathrm{e}_{\mathfrak{m}}(A)$.

Proposition 3.2. *Suppose $d = 1$ and $\overline{A}$ is a module-finite extension of A. Then the following four conditions are equivalent*:

(1) *Every $\mathfrak{m}$-primary ideal in A has minimal multiplicity.*

(2) *Either $\mathfrak{c} = A$ or $\mathfrak{c}$ has minimal multiplicity.*

(3) *Let I be an integrally closed $\mathfrak{m}$-primary ideal in A. Then $I = a\overline{A}$ for some $a \in I$.*

(4) *$\mathfrak{m} = f\overline{A}$ for some $f \in \mathfrak{m}$.*

When this is the case, the following hold true:

(i) $\mathrm{v}(A) = \mathrm{e}(A)$.

(ii) *Let I be an $\mathfrak{m}$-primary ideal in A. Then $I = \overline{I}$ if and only if $\mu_A(I) = \mathrm{e}(A)$.*

(iii) *Let I be an integrally closed $\mathfrak{m}$-primary ideal in A. Then $\overline{I}^n = I^n$ for all $n \geq 1$.*

Proof. The ring $\overline{A}$ is a PIR, as A is reduced (cf., e.g., [11, Proof of (3.6)]). We have $\overline{aA} = a\overline{A} \cap A$ for any regular element a of A.

(1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4): This is clear.

(2) $\Rightarrow$ (4): We may assume $\mathfrak{c} \neq A$. Let $\mathfrak{c} = a\overline{A}$ with $a \in \mathfrak{c}$. Then a is A-regular and $\mathfrak{c} = \overline{aA}$. Hence $\mathfrak{m}\mathfrak{c} = a\mathfrak{m}$, as the ideal $\mathfrak{c}$ has minimal multiplicity. Therefore $\mathfrak{m}\mathfrak{c}\overline{A} = \mathfrak{m}$ because $\mathfrak{c} = a\overline{A}$, whence $\mathfrak{m} = f\overline{A}$ for some $f \in \mathfrak{m}$.

(4) $\Rightarrow$ (1), (3), *and the last assertions*: Let I be an $\mathfrak{m}$-primary ideal in A and choose a minimal reduction $Q = aA$ of I. Then $\overline{I} = a\overline{A}$, because $\overline{I} = a\overline{A} \cap A$ and $a\overline{A} \subseteq \mathfrak{m}\overline{A} = \mathfrak{m}$. Hence $\mathfrak{m}\overline{I} = a\mathfrak{m}$ so that the ideal I has minimal multiplicity. Thus assertions (1) and (3) follow. We consider the last assertions. Note $\mathrm{v}(A) = \mathrm{e}(A)$, since $\mathfrak{m}$ has minimal multiplicity. On the other hand we have an isomorphism $\overline{I} \cong \overline{A}$, because $\overline{I} = a\overline{A}$. Taking $I = \mathfrak{m}$, this shows that all the A-modules $\mathfrak{m}, \overline{I}$, and $\overline{A}$ are isomorphic to each other. Hence we get $\mu_A(\overline{I}/I) = \mathrm{e}(A) - \mu_A(I)$, because $\mu_A(\mathfrak{m}) = \mathrm{e}(A)$ and $\mu_A(\overline{I}/I) = \mu_A(\overline{I}) - \mu_A(I)$ (recall that $\mathfrak{m}\overline{I} = \mathfrak{m}I$). Thus $I = \overline{I}$ if and only if $\mu_A(I) = \mathrm{e}(A)$. If $I = \overline{I}$, then we get $I^n = a^n\overline{A}$ for all $n \geq 1$ whence $I^n = \overline{I}^n$. This proves Proposition 3.2. □

Let us note an example of dimension 2.

Example 3.3. Let $A = k[[X^4, X^3Y, X^2Y^2, XY^3, Y^4]]$ be the subring of the formal power series ring $k[[X, Y]]$ over k in two variables X, Y. Let $I = (X^4, X^3Y, XY^3, Y^4)A$ and $Q = (X^4, Y^4)A$. Then

(1) $\mathfrak{m}^2 = I^2 = \mathfrak{m}I = \mathfrak{m}Q$. $I^2 \neq QI$ but $I^3 = QI^2$. The ideal I has minimal multiplicity with Q a minimal reduction.

(2) For this ideal $I, \mathscr{R}$ and $\mathscr{G}$ are Buchsbaum rings of $\mathrm{I}(\mathscr{R}) = \mathrm{I}(\mathscr{G}) = 2$.

(3) $\mathrm{H}^0_M(\mathscr{G}) = k$ and $\mathrm{H}^1_M(\mathscr{G}) = \mathrm{H}^1_M(\mathscr{R}) = k(-1)$. $\mathrm{H}^i_M(\mathscr{R}) = (0)$ if $i \neq 1, 3$.

(4) $\operatorname{depth} \mathscr{R} = \operatorname{depth} \mathscr{S} = \operatorname{depth}_{\mathscr{P}} \mathscr{C} = \operatorname{depth} \mathscr{G} + 1 = 1$.

Here $I(*)$ denotes the Buchsbaum invariant.

Proof. It is routine to check assertion (1). Because $(X^4t, Y^4t) \cdot X^2Y^2 \subseteq t^{-1}\mathscr{R}'$ but $X^2Y^2 \notin I$, we have $\operatorname{depth} \mathscr{G} = 0$. Therefore by Proposition 2.5 $\mathscr{S}$ is not a Cohen–Macaulay ring and so by Corollary 2.4 we see that $\operatorname{depth} \mathscr{R} = \operatorname{depth} \mathscr{S} = \operatorname{depth}_{\mathscr{P}} \mathscr{C} = \operatorname{depth} \mathscr{G} + 1 = 1$. Hence assertion (4) follows. To check assertions (2) and (3), we note that $\mathfrak{m}^i = I^i$ for all $i \neq 1$. Then because $\mathrm{R}(\mathfrak{m})/\mathscr{R} = [\mathrm{R}(\mathfrak{m})/\mathscr{R}]_1 = \mathfrak{m}t/It$ and $\mathfrak{m}t/It \cong k$, we get the exact sequence

$$0 \to \mathscr{R} \to \mathrm{R}(\mathfrak{m}) \to k(-1) \to 0$$

of graded $\mathscr{R}$-modules. Since $\mathrm{R}(\mathfrak{m})$ is a Cohen–Macaulay ring of dimension 3 (note that $\mathfrak{m}^2 = \mathfrak{m}Q$), applying the local cohomology functors $\mathrm{H}^i_M(*)$ $(i \in \mathbb{Z})$ to it, we see

$\mathrm{H}_M^1(\mathscr{R}) = k(-1)$ and $\mathrm{H}_M^i(\mathscr{R}) = (0)$ if $i \neq 1, 3$. Hence $\mathscr{R}$ is a Buchsbaum ring of $\mathrm{I}(\mathscr{R}) = 2$. Similarly, since $\mathfrak{m}^i = I^i$ for $i \neq I$, we have an exact sequence

$$0 \to k \to \mathscr{G} \to \mathrm{G}(\mathfrak{m}) \to k(-1) \to 0$$

of graded $\mathscr{R}$-modules, from which we have $\mathrm{H}_M^0(\mathscr{G}) = k$ and $\mathrm{H}_M^1(\mathscr{G}) = k(-1)$. Hence the ring $\mathscr{G}$ is a quasi-Buchsbaum ring of $\mathrm{I}(\mathscr{G}) = 2$. To check that $\mathscr{G}$ is actually a Buchsbaum ring, we look at the exact sequences

(a) $0 \to \mathscr{G} \to \mathrm{R}(\mathfrak{m})/I\mathrm{R}(\mathfrak{m}) \to k(-1) \to 0$ and

(b) $0 \to k \to \mathrm{R}(\mathfrak{m})/I\mathrm{R}(\mathfrak{m}) \to \mathrm{G}(\mathfrak{m}) \to 0$.

Then by (b) we have $\mathrm{H}_M^1(\mathrm{R}(\mathfrak{m})/I\mathrm{R}(\mathfrak{m})) = (0)$, because $\mathrm{G}(\mathfrak{m})$ is a Cohen–Macaulay ring of dimension 2. Therefore from (a) the canonical epimorphism

$$k(-1) \to \mathrm{H}_M^1(\mathscr{G}) \to \mathrm{H}_M^1(\mathrm{R}(\mathfrak{m})/I\mathrm{R}(\mathfrak{m})) = (0)$$

follows. Hence the subjectivity criterion for Buchsbaumness [18, Chapter I, (2.16)] can be applied and so $\mathscr{G}$ is a Buchsbaum ring. $\square$

From this example one might guess that $\operatorname{Proj}\mathscr{R}$ and $\operatorname{Proj}\mathscr{G}$ are always Cohen–Macaulay schemes for the $\mathfrak{m}$-primary ideals of minimal multiplicity. This is not true in general. Here let us note one example. Firstly recall that the scheme $\operatorname{Proj}\mathscr{R}$ (resp. $\operatorname{Proj}\mathscr{G}$) is Cohen–Macaulay if and only if $\mathrm{H}_M^i(\mathscr{R})$ (resp. $\mathrm{H}_M^i(\mathscr{G})$) is a finitely generated $\mathscr{R}$-module (resp. a finitely generated $\mathscr{G}$-module) for all $i \neq d+1$ (resp. $i \neq d$) (cf. [5, (2.5), (2.11), and (3.8)]; note $\sqrt{I} = \mathfrak{m}$). The scheme $\operatorname{Proj}\mathscr{R}$ is Cohen–Macaulay if and only if $\operatorname{Proj} G$ is also Cohen–Macaulay. And when this is the case, the sequence $b_1t, b_2t, \ldots, b_st$ ($s = \operatorname{depth}\mathscr{G}$) is $\mathscr{G}$-regular for any system $b_1, b_2, \ldots, b_d$ of generators for Q (cf. [5, (2.5) and (2.11)]).

Example 3.4. Let $A = k[[X^2, Y^2, Z^2, XY, YZ, ZX]]$ in the formal power series ring $k[[X, Y, Z]]$. Let $I = (X^2, Y^2, Z^2, XY, YZ)A$ and $Q = (X^2, Y^2, Z^2)A$. Then $I^3 = QI^2$ and $\mathfrak{m}I = \mathfrak{m}Q$. The ring $\mathscr{S}$ is Cohen–Macaulay and $\operatorname{depth}\mathscr{R} = \operatorname{depth}\mathscr{G} + 1 = 3$. Hence $\mathscr{R}$ is not Cohen–Macaulay. The scheme $\operatorname{Proj}\mathscr{R}$ is not Cohen–Macaulay, because $\mathrm{H}_M^2(\mathscr{G})$ is not a finitely generated $\mathscr{G}$-module.

Proof. It is routine to check that $I^3 = QI^2$ and $\mathfrak{m}I \subseteq Q$. We will show that X^2t, Z^2t form a $\mathscr{G}$-regular sequence. It is enough to see $(X^2, Z^2) \cap I^n = (X^2, Z^2)I^{n-1}$ for all $n \geq 2$ (cf. [19, (2.7)]). Note $I^2 = (X^2, Z^2)I + Y^2\mathfrak{m}$. Then we have $(X^2, Z^2) \cap I^2 = (X^2, Z^2)I + (X^2, Z^2) \cap Y^2\mathfrak{m}$, whence $(X^2, Z^2) \cap I^2 = (X^2, Z^2)I$ as $(X^2, Z^2) \cap Y^2\mathfrak{m} \subseteq Y^2(X^2, Z^2)$. Let $n \geq 3$ and assume that our equality holds true for $n-1$. Then $I^n = QI^{n-1}$ so that $(X^2, Z^2) \cap I^n = (X^2, Z^2)I^{n-1} + (X^2, Z^2) \cap Y^2 I^{n-1}$. As $(X^2, Z^2) \cap Y^2 I^{n-1} = Y^2[(X^2, Z^2) \cap I^{n-1}]$, from the hypothesis on n we see $(X^2, Z^2) \cap Y^2 I^{n-1} = Y^2(X^2, Z^2)I^{n-2}$. Thus $(X^2, Z^2) \cap I^n = (X^2, Z^2)I^{n-1}$ so that $\operatorname{depth}\mathscr{G} \geq 2$. Hence by (2.7) $\mathscr{S}$ is a Cohen–Macaulay ring. As $XY^2Z \in I^2$ but $XZ \notin I$, Y^2t is a zerodivisor in $\mathscr{G}$. Therefore $\operatorname{depth}\mathscr{G} = 2$ and $\operatorname{depth}\mathscr{R} = 3$ by Lemma 2.3(3). If $\mathrm{H}_M^2(\mathscr{G})$ were a finitely generated

$\mathscr{G}$-module, every subsystem f, g of homogeneous parameters for $\mathscr{G}$ would be a $\mathscr{G}$-regular sequence, which is impossible because Y^2t is a zerodivisor in $\mathscr{G}$. Since the finite generation of $\mathrm{H}^2_M(\mathscr{G})$ is equivalent to the Cohen–Macaulayness of $\operatorname{Proj}\mathscr{G}$, we have that $\operatorname{Proj}\mathscr{G}$ cannot be Cohen–Macaulay. Hence $\operatorname{Proj}\mathscr{R}$ is not Cohen–Macaulay as well. $\square$

We close this section with the following example.

Example 3.5. Let $n \geq d \geq 2$ be integers and let $R = k[X_1, X_2, \ldots, X_d]$ be the polynomial ring in d variables over k. let $S = R^{(n)}$ denote the Veronesean subring of R with order n. We put $\mathfrak{M} = S_+$ and $A = S_{\mathfrak{M}}$. Let $Q = (X_1^n, X_2^n, \ldots, X_d^n)A$ and $I = Q + WA$, where $W = S_{d-1} = R_{n(d-1)}$. Then $I^2 = QI$ and $\mathfrak{m}I = \mathfrak{m}Q$. The ring $\mathscr{R}$ is Cohen–Macaulay,

$$d = \dim A \quad \text{and} \quad \mu_A(I) = d + \binom{n-1}{d-1}.$$

Proof. The ring S is Cohen–Macaulay with $\dim S = d$ and $\mathrm{a}(S) = -1$ (cf. [9, (3.1.1)]). Hence $[S/(X_1^n, X_2^n, \ldots, X_d^n)S]_{d-1} \neq (0)$ but $[S/(X_1^n, X_2^n, \ldots, X_d^n)S]_i = (0)$ for $i \geq d$. Therefore $I^2 = QI$ and $\mathfrak{m}I \subseteq Q$ so that by Theorem 2.7 $\mathscr{R}$ is a Cohen–Macaulay ring. We get

$$\mu_A(I) = d + \binom{n-1}{d-1},$$

because $\mu_A(I) = d + \mu_A(I/Q)$ and $\mu_A(I/Q) = \dim_k [S/(X_1^n, X_2^n, \ldots, X_d^n)S]_{d-1} = \dim_k [R/(X_1^n, X_2^n, \ldots, X_d^n)R]_{n(d-1)}$. $\square$

4. Cohen–Macaulayness in Proj R(I) versus the negativity of $\mathrm{a}_i(\mathrm{R}(I))$'s

As explored in [14], at least in the case where $\dim A$ is small, the negativity of $\mathrm{a}_i(\mathrm{R}(I))$'s has some influence on the Cohen–Macaulayness in $\mathrm{R}(I)$. We shall also discuss this phenomenon in our context. We maintain the same notation as given in Section 2. To begin with, we note the following.

Proposition 4.1. *Suppose that $I^{n+1} = \mathfrak{m}I^n$ for some $n \geq 0$. Then* $\operatorname{Proj}\mathscr{R}$ *is a Cohen–Macaulay scheme.*

Proof. By [13, (2.13)] it suffices to check that $a_1, a_2, \ldots, a_d$ is a d-sequence on I^p for all $p \geq n+2$. Let $Q_i = (a_1, a_2, \ldots, a_i)$ for $0 \leq i \leq d$. Firstly we will show that $Q_i \cap I^p = Q_i I^{p-1}$. In fact, as $I^p = \mathfrak{m}I^{p-1} = \mathfrak{m}Q^{p-1}$ and $Q_i \cap Q^{p-1} = Q_i Q^{p-2}$, we see $Q_i \cap I^p \subseteq Q_i Q^{p-2} \cap \mathfrak{m}Q^{p-1} \subseteq \mathfrak{m}Q_i \cdot Q^{p-2} = Q_i I^{p-1}$ (note that $\mathrm{G}(Q)$ is a polynomial ring). Hence $Q_i \cap I^p = Q_i I^{p-1}$ for $0 \leq i \leq d$. Let $1 \leq i \leq j \leq d$ be integers and choose $x \in I^p$ so that $a_i a_j x \in Q_{i-1} I^p$. Then $x \in Q_{i-1} \cap I^p = Q_{i-1} I^{p-1}$ whence $a_j x \in Q_{i-1} I^p$, and thus $a_1, a_2, \ldots, a_d$ is a d-sequence on I^p. $\square$

Example 4.2. Let k be an infinite field. Let $A = k[[X^5, X^4Y, X^3Y^2, X^2Y^3, XY^4, Y^5]]$ in $k[[X,Y]]$, where $k[[X,Y]]$ is the formal power series ring in two variables. Let $I = (X^5, X^4Y, XY^4, Y^5)A$ and $Q = (X^5, Y^5)A$. Then $I^4 = QI^3$ and $\mathfrak{m}I = \mathfrak{m}Q$. The ring $\mathscr{R}$ is not Cohen–Macaulay by (2.9) since $I^3 \neq QI^2$, while $\operatorname{Proj}\mathscr{R}$ is a Cohen–Macaulay scheme by Proposition 4.1 because $I^3 = \mathfrak{m}I^2$. As $X^3Y^2 \notin I$ but $(X^{10}, Y^{10}) \cdot X^3Y^3 \subseteq I^3$, we have $\operatorname{depth}\mathscr{G} = 0$. Hence $\operatorname{depth}\mathscr{R} = \operatorname{depth}\mathscr{S} = 1$ by Corollary 2.4 and Proposition 2.5.

Let N be the unique graded maximal ideal in $\mathrm{R}(Q)$. Recall that $\mathrm{a}(\mathscr{R}) = -1$ (cf. [7, Part II, Eq. (3.3)]).

Lemma 4.3. (1) $\mathrm{H}^1_M(\mathscr{R})$ *is a finitely generated* $\mathscr{R}$*-module and* $\mathfrak{m} \cdot \mathrm{H}^1_M(\mathscr{R}) = (0)$.
(2) $\mathrm{H}^0_N(\mathscr{C}) = \mathrm{H}^1_N(\mathscr{C}) = (0)$ *if* $\mathrm{a}_1(\mathscr{R}) < 0$.
(3) $I = Q$ *if* $d = 1$ *and* $\mathrm{a}_1(\mathscr{R}) < 0$.

Proof. (1) The second assertion follows from the embedding $\mathrm{H}^1_M(\mathscr{R}) \subseteq \mathrm{H}^1_N(\mathscr{C})$ (cf. (2.2)). To see the first one we may assume A is complete. Let $\mathrm{K}_{\mathscr{R}}$ be the graded canonical module of $\mathscr{R}$. Then $(0) :_{\mathscr{R}} \mathrm{K}_{\mathscr{R}} = (0)$ as $\dim \mathscr{R}/P = d+1$ for all $P \in \operatorname{Ass}\mathscr{R}$ (cf. [20, (1.7)]). Let $E = \operatorname{End}_{\mathscr{R}} \mathrm{K}_{\mathscr{R}}$ and apply the functors $\mathrm{H}^i_M(*)$ to the exact sequence $0 \to \mathscr{R} \to E \to E/\mathscr{R} \to 0$. Then $\mathrm{H}^0_M(E/\mathscr{R}) \cong \mathrm{H}^1_M(\mathscr{R})$ as $\operatorname{depth}_{\mathscr{R}} E \geq 2$. Thus $\mathrm{H}^1_M(\mathscr{R})$ is a finitely generated $\mathscr{R}$-module.

(2) *and* (3): Let $f = a_d t$. Then f is a nonzerodivisor on $\mathscr{C}$ (cf. Proof of Lemma 2.3(1)). Let $\overline{\mathscr{C}} = \mathscr{C}/f\mathscr{C}$. Then by the exact sequence $0 \to \mathscr{C}(-1) \xrightarrow{f} \mathscr{C} \to \overline{\mathscr{C}} \to 0$, we get the embedding $\mathrm{H}^0_N(\overline{\mathscr{C}}) \subseteq \mathrm{H}^1_N(\mathscr{C})(-1)$. Hence $\mathrm{a}_0(\overline{\mathscr{C}}) \leq \mathrm{a}_1(\mathscr{C}) + 1$. Note $\mathrm{a}_1(\mathscr{C}) = \mathrm{a}_1(\mathscr{R})$ (resp. $\mathrm{a}_1(\mathscr{C}) \leq \max\{\mathrm{a}_1(\mathscr{R}), -1\}$) if $d \geq 2$ (resp. $d = 1$) (cf. Proposition 2.2). And we see $\mathrm{a}_1(\mathscr{C}) \leq -1$ so that $\mathrm{a}_0(\overline{\mathscr{C}}) \leq 0$. Because $\overline{\mathscr{C}}_n = (0)$ for $n \leq 0$, this forces $\mathrm{H}^0_N(\overline{\mathscr{C}}) = (0)$ whence $\mathrm{H}^1_N(\mathscr{C}) = (0)$. Assertion (3) is clear. □

Firstly we note the following result in the case where $\dim A = 2$.

Proposition 4.4. *Suppose* $d = 2$. *Then*
(1) $\mathrm{H}^1_M(\mathscr{S})$ *is a finitely generated* $\mathscr{S}$-module.
(2) $\mathscr{S}$ *is a Cohen–Macaulay ring if* $\mathrm{a}_1(\mathscr{R}) < 0$.
(3) $\mathscr{R}$ *is a Cohen–Macaulay ring if and only if* $\mathrm{a}_i(\mathscr{R}) < 0$ *for all* $i \in \mathbb{Z}$.

Proof. (1) This follows from Proposition 2.2(1) and Lemma 4.3(1).

(2) See Proposition 2.2(1). Note that by Lemma 4.3(2) $\mathrm{H}^i_N(\mathscr{C}) = (0)$ for $i \leq 1$.

(3) Assume $\mathrm{a}_i(\mathscr{R}) < 0$ for all $i \in \mathbb{Z}$. Then $\mathscr{S}$ is Cohen–Macaulay by (2). We have $\mathrm{a}(\mathscr{S}) \leq -1 = 1 - d$, because $\mathrm{a}(\mathscr{S}) = \max\{\mathrm{a}_2(\mathscr{C}), -2\}$ and $\mathrm{a}_2(\mathscr{C}) \leq \max\{\mathrm{a}_2(\mathscr{R}), -1\}$ by parts (3) and (4) of Proposition 2.2. Hence $\mathscr{R}$ is a Cohen–Macaulay ring by Corollary 2.6 and Theorem 2.7. □

Proposition 4.5. *Suppose* $d = 3$. *Then* $\mathscr{S}$ *is a Cohen–Macaulay ring and* $I^3 = QI^2$ *if and only if* $\mathrm{a}_i(\mathscr{R}) < 0$ *for all* $i \in \mathbb{Z}$.

Proof. Assume that $\mathscr{S}$ is a Cohen–Macaulay ring and $I^3 = QI^2$. Then $\mathrm{a}(\mathscr{S}) \leq -1$ by Proposition 2.5 so that $\mathrm{a}_3(\mathscr{R}) \leq -1$ by (3) and (4) of Proposition 2.2. Hence $\mathrm{a}_i(\mathscr{R}) < 0$ for all $i \in \mathbb{Z}$ (recall that $\mathrm{a}(\mathscr{R}) = -1$ and $\operatorname{depth} \mathscr{R} \geq 3$ by Proposition 2.2(1)). Conversely assume that $\mathrm{a}_i(\mathscr{R}) < 0$ for all $i \in \mathbb{Z}$. Firstly we will show that $\mathscr{S}$ is a Cohen–Macaulay ring. Assume the contrary. Then $\operatorname{depth} \mathscr{S} = 2$ by Proposition 2.2(1) because $\operatorname{depth}_{\mathscr{P}} \mathscr{C} \geq 2$ by Lemma 4.3(2). Hence $\operatorname{depth} \mathscr{R} = 2$ and $\operatorname{depth} \mathscr{G} = 1$ by Corollary 2.4. Let $\mathfrak{a} = \mathscr{R}_+$ and consider the standard exact sequences

(a) $0 \to \mathfrak{a} \to \mathscr{R} \to A \to 0$ and

(b) $0 \to \mathfrak{a}(1) \to \mathscr{R} \to \mathscr{G} \to 0$.

Then applying the functors $\mathrm{H}^i_M(*)$ to (a) and (b), we get an isomorphism $\mathrm{H}^2_M(\mathfrak{a}) \cong \mathrm{H}^2_M(\mathscr{R})$ and the embedding $\mathrm{H}^1_M(\mathscr{G}) \subseteq \mathrm{H}^2_M(\mathfrak{a})(1)$. Hence $\mathrm{a}_1(\mathscr{G}) \leq -2$ because $\mathrm{a}_2(\mathfrak{a}) = \mathrm{a}_2(\mathscr{R}) < 0$ and $\mathrm{a}_1(\mathscr{R}) \leq \mathrm{a}_2(\mathfrak{a}) - 1$. Choose an element $g \in \mathscr{G}_1$ so that g is $\mathscr{G}$-regular (this choice is possible, because $\operatorname{depth} \mathscr{G} > 0$ and $A/\mathfrak{m}$ is infinite). Let $\overline{\mathscr{G}} = \mathscr{G}/g\mathscr{G}$ and apply $\mathrm{H}^i_M(*)$ to the exact sequence $0 \to \mathscr{G}(-1) \xrightarrow{g} \mathscr{G} \to \overline{\mathscr{G}} \to 0$. Then from the embedding $\mathrm{H}^0_M(\overline{\mathscr{G}}) \subseteq \mathrm{H}^1_M(\mathscr{G})(-1)$ we see $\mathrm{a}_0(\overline{\mathscr{G}}) \leq \mathrm{a}_1(\mathscr{G}) + 1$; hence $\mathrm{a}_0(\overline{\mathscr{G}}) \leq -1$. Therefore $\mathrm{H}^0_M(\overline{\mathscr{G}}) = (0)$ so that $\operatorname{depth} \mathscr{G} \geq 2$, which is absurd and thus $\mathscr{S}$ is a Cohen–Macaulay ring. Because $\mathrm{a}(\mathscr{S}) = \max\{\mathrm{a}_3(\mathscr{C}), -3\}$ and $\mathrm{a}_3(\mathscr{C}) \leq \max\{\mathrm{a}_3(\mathscr{R}), -1\}$ by Proposition 2.2, we get $\mathrm{a}(\mathscr{S}) \leq -1$ whence $\mathrm{r}_Q(I) \leq 2$ by Proposition 2.5. Thus $I^3 = QI^2$, which completes the proof of Proposition 4.5. □

Let us note the following sufficient condition for the Rees algebras $\mathscr{R}$ to be Cohen–Macaulay rings.

Theorem 4.6. *Suppose $\mu_A(I) \geq \mathrm{r}(A) + 1$ or $\mathrm{r}(A/I) \leq d - 1$. Assume that $\operatorname{Proj} \mathscr{R}$ is a Cohen–Macaulay scheme and $\mathscr{S}$ is a Cohen–Macaulay ring. Then $I^2 = QI$ and so $\mathscr{R}$ is a Cohen–Macaulay ring if $d \geq 2$.*

Proof. Let $x \in I^2$ and write $x = \sum_{i=1}^d a_i x_i$ with $x_i \in A$. Then for the same reason as is in the proof of Corollary 2.8, we get $x_i \in Q : \mathfrak{m}$ for all $1 \leq i \leq d$. Let $J = Q : \mathfrak{m}$. Then $\mathrm{r}(A/I) \geq \ell_A(J/I) = \ell_A(J/Q) - \ell_A(I/Q) = \mathrm{r}(A) - \mu_A(I) + d$, because $\ell_A(J/Q) = \mathrm{r}(A)$ and $\ell_A(I/Q) = \mu_A(I) - d$. Therefore if $\mathrm{r}(A/I) \leq d - 1$ or more generally $\mu_A(I) \geq \mathrm{r}(A) + 1$, we have $\ell_A(J/I) \leq d - 1$ so that the elements $x_1, x_2, \ldots, x_d \bmod I$ cannot be $A/\mathfrak{m}$-linearly independent in J/I. Without loss of generality we may write $x_d = \sum_{i=1}^{d-1} c_i x_i + y$ with $c_i \in A$ and $y \in I$. Then since $x = \sum_{i=1}^d a_i x_i = \sum_{i=1}^{d-1} (a_i + a_d c_i) x_i + a_d y$, we have $x - a_d y \in (a_i + a_d c_i \mid 1 \leq i \leq d - 1) \cap I^2$. Recall that $\operatorname{depth} \mathscr{G} \geq d - 1$ by Proposition 2.5 since $\mathscr{S}$ is a Cohen–Macaulay ring. And we get $(a_i + a_d c_i \mid 1 \leq i \leq d - 1) \cap I^2 = (a_i + a_d c_i \mid 1 \leq i \leq d - 1)I$ by [19, 2.7], because $Q = (a_i + a_d c_i \mid 1 \leq i \leq d - 1) + (a_d)$ and because $\operatorname{Proj} \mathscr{G}$ is a Cohen–Macaulay scheme with $\operatorname{depth} \mathscr{G} \geq d - 1$. Hence $x - a_d y \in (a_i + a_d c_i \mid 1 \leq i \leq d - 1)I \subseteq QI$ so that $x \in QI$. Thus $I^2 = QI$. □

Corollary 4.7. *Suppose $\mathrm{r}(A) \leq d$. Then $I^2 = QI$ if $\operatorname{Proj} \mathscr{R}$ is a Cohen–Macaulay scheme and $\mathscr{S}$ is a Cohen–Macaulay ring.*

Proof. We may assume $I \neq Q$. Hence $\mu_A(I) \geq d+1$ and the assertion follows from Theorem 4.6. $\square$

Corollary 4.8. *Suppose $d=3$ and $\mathrm{r}(A) \leq 3$. Then $\mathscr{R}$ is a Cohen–Macaulay ring if and only if $\operatorname{Proj}\mathscr{R}$ is a Cohen–Macaulay scheme and $\mathrm{a}_i(\mathscr{R}) < 0$ for all $i \in \mathbb{Z}$*

Proof. See Theorem 2.7, Proposition 4.5 and Corollary 4.7. $\square$

5. A counterexample

Let k be an algebraically closed field. Let $R = k[X,Y,Z,V,A,B,C]$ be the polynomial ring in 7 variables over k and let

$$\mathfrak{a} = (X,Y,Z)\cdot(X,Y,Z,V) + (V^2 - (AX+BY+CZ)).$$

We put $S = R/\mathfrak{a}$ and let $x,y,z,\ldots,c$ denote respectively the reductions of $X,Y,Z,\ldots,C \bmod \mathfrak{a}$. Let $M = S_+$, $\mathcal{O} = S_M$, and $\mathfrak{m} = MS_M$. We put $Q = (a,b,c)\mathcal{O}$ and $I = Q + v\mathcal{O}$.

Example 5.1. The following assertions hold true:
(1) $(\mathcal{O},\mathfrak{m})$ is a Cohen–Macaulay local ring of $\dim \mathcal{O} = 3$.
(2) $\mathfrak{m}^2 = \mathfrak{m}I = \mathfrak{m}Q$, $I^3 = QI^2$ but $I^2 \neq QI$. Hence the rings $\mathrm{R}(I)$ and $\mathrm{G}(I)$ are not Cohen–Macaulay.
(3) $\mathrm{e}_{\mathfrak{m}}(\mathcal{O}) = 5$ and $\mathrm{r}(\mathcal{O}) = 4$.
(4) $\operatorname{depth}\mathrm{R}(I) = 3$ and $\mathrm{a}_3(\mathrm{R}(I)) < 0$. Hence $\mathrm{a}_i(\mathrm{R}(I)) < 0$ for all $i \in \mathbb{Z}$.
(5) $\operatorname{Proj}\mathrm{R}(I)$ is a Cohen–Macaulay scheme.

Proof. (1)–(3): Let $\mathfrak{q} = (a,b,c)S$ and $J = \mathfrak{q} + vS$. Hence $M = J + (x,y,z)S$ and $MJ = M\mathfrak{q}$. Let $P = (X,Y,Z,V)R$. Then $P = \sqrt{\mathfrak{a}}$ so that $\dim \mathcal{O} = \dim S = 3$. Since $v^2 = ax + by + cz$ and $v^3 = 0$, we get $M^2 = \mathfrak{q}M$ and $J^3 = \mathfrak{q}J^2$. Therefore, a,b,c form a homogeneous system of parameters for S with $S/\mathfrak{q} \cong k[X,Y,Z,V]/(X,Y,Z,V)^2$, whence $\ell_S(S/\mathfrak{q}) = 5$. Consequently, to see that S is a Cohen–Macaulay ring, it suffices to prove the following claim.

Claim 5.2. $\mathrm{e}_{\mathfrak{q}}(S) = 5$.

Proof. We have $\mathrm{e}_{\mathfrak{q}}(S) = \ell_{R_P}(R_P/\mathfrak{a}R_P)$, because $P = \sqrt{\mathfrak{a}}$ and $R/P \cong k[A,B,C]$. Let $\tilde{k} = k[C, 1/C]$ and $\tilde{R} = R[1/C]$. Then $\tilde{R} = \tilde{k}[X_1,Y_1,Z_1,V_1,A_1,B_1]$ where $X_1 = X/C$, $Y_1 = Y/C$, $Z_1 = Z/C,\ldots,$ and $B_1 = B/C$. As $\mathfrak{a}\tilde{R} = (X_1,Y_1,Z_1)\cdot(X_1,Y_1,Z_1,V_1) + (V_1^2 - (A_1X_1 + B_1Y_1) - Z_1)$ and as X_1,Y_1,Z_1,V_1,A_1 and B_1 are algebraically independent over $\tilde{k}$, substituting Z_1 with $V_1^2 - (A_1X_1 + B_1Y_1)$, we get the identification

$$\tilde{R}/\mathfrak{a}\tilde{R} \cong \tilde{k}[X_1,Y_1,V_1,A_1,B_1]/(X_1,Y_1,V_1)(X_1,Y_1,V_1^2).$$

Let T denote the ring of the right-hand side. Then the ideal $P\tilde{R}/\mathfrak{a}\tilde{R}$ corresponds, via the identification, to the prime ideal $\mathfrak{p} = (X_1,Y_1,V_1)T$ so that, counting the number of surviving monomials in X_1, Y_1, and V_1, we readily get $\ell_{R_P}(R_P/\mathfrak{a}\, R_P) = \ell_{T_\mathfrak{p}}(T_\mathfrak{p}) = 5$. Hence $\mathrm{e}_\mathfrak{q}(S) = 5$ and S is a Cohen–Macaulay ring. $\square$

Suppose $v^2 \in \mathfrak{q}J$ and write $v^2 = av_1 + bv_2 + c_3v_3$ with $v_i \in J$. Then since $ax + by + cz = av_1 + bv_2 + c_3v_3$ and since a, b, c is an S-regular sequence, we have $z - v_3 \in (a,b)S$. Consequently $Z \in (A,B,C,V)R + \mathfrak{a}$, which is impossible because the ideal $\mathfrak{a}$ is generated by forms of degree 2. Hence $v^2 \notin \mathfrak{q}J$ so that we have $I^2 \neq QI$. Therefore by Theorem 2.7 the rings $\mathrm{R}(I)$ and $\mathrm{G}(I)$ cannot be Cohen–Macaulay. As $\mathfrak{m}^2 = Q\mathfrak{m}$, we get $\mathrm{r}(\mathcal{O}) = \mathrm{r}(\mathcal{O}/Q) = \ell_\mathcal{O}(\mathfrak{m}/Q) = 4$. Of course $\mathrm{e}_\mathfrak{m}(\mathcal{O}) = \mathrm{e}_Q(\mathcal{O}) = 5$ by Claim 5.2.

(4) We need the following.

Claim 5.3. *$aS \cap J^n = aJ^{n-1}$ and $(a,b)S \cap J^n = (a,b)J^{n-1}$ for all $n \in \mathbb{Z}$.*

Proof. We may assume $n \geq 2$. Firstly we will check the second equality. Since $J^2 = \mathfrak{q}J + v^2S$, we have $(a,b)S \cap J^2 = (a,b)J + (a,b)S \cap (cJ + v^2S)$. Let $\varphi \in (a,b)S \cap (cJ + v^2S)$ and write $\varphi = ci + v^2\xi$ with $i \in J$ and $\xi \in S$. Then because $v^2 = ax + by + cz$ and $\varphi \in (a,b)S$, we see $c(i + z\xi) \in (a,b)S$ so that $i + z\xi \in (a,b)S \subseteq J$. Hence $z\xi \in J$. As $z \notin J$, this forces $\xi \in M = J + (x,y,z)S$. Let $\xi = j + (\alpha x + \beta y + \gamma z)$ with $j \in J$ and $\alpha, \beta, \gamma \in S$. Then $\varphi = a(xj) + b(yj) + c(i + zj)$ because $v^2\xi = (ax + by + cz)j$. Consequently $i + zj \in (a,b)S$ as $\varphi \in (a,b)S$, whence $\varphi \in (a,b)J$. Thus $(a,b)S \cap (cJ + v^2S) \subseteq (a,b)J$ so that we have $(a,b)S \cap J^2 = (a,b)J$. Now let $n \geq 3$ and suppose that $(a,b)S \cap J^{n-1} = (a,b)J^{n-2}$. Then because $J^n = \mathfrak{q}J^{n-1}$, we see

$$\begin{aligned}
(a,b)S \cap J^n &= (a,b)J^{n-1} + (a,b)S \cap cJ^{n-1} \\
&= (a,b)J^{n-1} + c[(a,b)S \cap J^{n-1}] \\
&= (a,b)J^{n-1} + c \cdot (a,b)J^{n-2} \quad \text{(by the hypothesis on } n) \\
&= (a,b)J^{n-1}.
\end{aligned}$$

This proves the second equality. The first one easily follows, by induction on n, from the second. $\square$

By Claim 5.3 and [19, (2.7)] we get depth $\mathrm{G}(I) = 2$, since by (2) $\mathrm{G}(I)$ is not Cohen–Macaulay. Therefore depth $\mathrm{R}(I) = 3$ by Lemma 2.3(3). On the other hand, by Proposition 2.5 is a Cohen–Macaulay ring of $\mathrm{a}(\mathrm{S}(I)) = \mathrm{r}_Q(I) - 3 = -1$. Hence $\mathrm{a}_3(\mathrm{R}(I)) < 0$ by Proposition 2.2(3) and (4) so that $a_i(\mathrm{R}(I)) < 0$ for all $i \in \mathbb{Z}$.

(5) By [13, (2.13)] this follows from Claim 5.3 (recall that the field $k = \mathcal{O}/\mathfrak{m}$ is algebraically closed). $\square$

Remark 5.4. Example 5.1 shows the assumption in Corollaries 4.7 and 4.8 that $\mathrm{r}(A) \leq d$ is not superfluous. It provides with a counterexample the main conjecture explored by [15]. By [6, (9.5)] we get $\mathrm{H}^3_M(\mathscr{R}) = k(1)$. Hence the Rees algebras $\mathrm{R}(I^n)$ and the multi Rees algebras $\mathrm{R}(I;n)$ of n copies of the ideal I are Cohen–Macaulay rings for all integers $n \geq 2$. We have $\mathrm{a}_3(\mathscr{R}) = \mathrm{a}_4(\mathscr{R}) = -1$. The ring $\mathcal{O}$ is a Cohen–Macaulay local ring of maximal embedding dimension, while $\mathcal{O}_\mathfrak{p}$ does not have maximal embedding dimension since $p^2\mathcal{O}_p \neq (0)$ (here $\mathfrak{p}$ denotes the unique minimal prime ideal in $\mathcal{O}$). By this example we see that local rings of a Cohen–Macaulay local ring possessing maximal embedding dimension do not always have maximal embedding dimension.

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